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GROUP RINGS WITH SOLVABLE UNIT GROUPS

Dedicated to Professor TAKESHI INAGAKI on the occasion
of his sixtieth birthday

KAORU MOTOSE and HISAO TOMINAGA

Throughout, R will represent an artinian simple ring with 1, C the center of R, and G a finite group of order g.

Recently, in their paper [1], P. B. Bhattacharya and S. K. Jain proved the following : Let S be a right artinian ring with 1, and $J(S)$ the (Jacobson) radical of S. Then, the unit group $U(S)$ of S is solvable if and only if $U(S/J(S))$ is solvable[1], namely, $S/J(S)$ is a finite direct sum of fields, $(\mathrm{GF}(2))_2$ and $(\mathrm{GF}(3))_2$. This result will be used freely in the sequel. The main theme of our discussion will concern the solvability of the unit group of the group ring RG.

For semi-simple group rings, we shall prove the following :

Theorem 1. *Assume that g is not divisible by the characteristic of C. Then, $U(RG)$ is solvable if and only if there holds one of the following*:

(i) *$R=C$ and G is abelian (or equivalently, $U(RG)$ is nilpotent[2]).*

(ii) *$R=(\mathrm{GF}(2))_2$ and $G=1$.*

(iii) *$R=(\mathrm{GF}(3))_2$ and G is an abelian group of exponent 2.*

(iv) *$R=\mathrm{GF}(3)$ and $G=G_1\times G_2$, where G_1 is an abelian group of exponent 2 and G_2 is a non-abelian indecomposable 2-group such that $U(RG_2)$ is solvable.*

Proof. Assume first $U(RG)$ is solvable. Since $U(R)$ is solvable, $R=C$ or $(\mathrm{GF}(2))_2$ or $(\mathrm{GF}(3))_2$. Now, we shall distinguish between three cases :

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1) More generally, there holds the following : Let S be a ring with 1, and N a nilpotent ideal of S. If $U(S/N)$ is solvable then $U(S)$ is solvable (and conversely). To see this, we consider the group homomorphism $\alpha: U(S)\to U(S/N)$ induced by the natural ring homomorphism $S\to S/N$. Let $(\mathrm{Ker}\,\alpha)^{[n]}$ be the n-th commutator subgroup of $\mathrm{Ker}\,\alpha$. Noting that $a^{-1}b^{-1}ab-1=a^{-1}b^{-1}\{(a-1)(b-1)-(b-1)(a-1)\}$ for $a,\ b\in U(S)$, we can easily see $\{u-1\mid u\in(\mathrm{Ker}\ \alpha)^{[n]}\}\subset N^{2^n}$, which implies the solvability of $\mathrm{Ker}\,\alpha$.

2) Cf. [3].

Case 1. $R=C\neq \mathrm{GF}(3)$: It suffices then to consider the case $R=\mathrm{GF}(2)$: $RG=F_1\oplus\cdots\oplus F_k\oplus(\mathrm{GF}(2))_2^{(l)}$, where F_i are fields over GF(2) and $(\mathrm{GF}(2))_2^{(l)}$ means the direct sum of l copies of $(\mathrm{GF}(2))_2$. Since GL(2, 2) is isomorphic to the symmetric group S_3 and the alternative group A_3 is abelian, we obtain a normal chain $G=G_0\triangleright G_1\triangleright\cdots\triangleright G_{n-1}\triangleright G_n$ such that $(G_{i-1}:G_i)\leqslant 2$ and G_n is abelian. Noting that g is odd, we readily see that G is equal to the abelian group G_n.

Case 2. $R=(\mathrm{GF}(2))_2$ or $(\mathrm{GF}(3))_2$: Evidently, $RG=F_1\oplus\cdots\oplus F_k\oplus R^{(l)}$. Since $U(CG)$ is solvable and $RG=(CG)_2$, one will easily see that $CG=C^{(g)}$. If $R=(\mathrm{GF}(2))_2$ then $U(CG)=1$, and hence $G=1$. On the other hand, if $R=(\mathrm{GF}(3))_2$ then $U(CG)$ is an abelian group of exponent 2, and hence so is G.

Case 3. $R=\mathrm{GF}(3)$: Assume G is non-abelian. Then, $G=G_1\times G_2$, where G_2 is non-abelian and indecomposable. Since both $U(RG_1)$ and $U(RG_2)$ are solvable, $RG_1=F_1'\oplus\cdots\oplus F_{k'}'\oplus(R)_2^{(l')}$ and $RG_2=F_1''\oplus\cdots\oplus F_{k''}''\oplus(R)_2^{(l'')}$ where F_i' and F_j'' are fields over R and $l''>0$. Accordingly, RG is the direct sum of $\oplus_{i,j}(F_i'\otimes_R F_j'')$, $\oplus_i(F_i')_2^{(l'')}$, $\oplus_j(F_j'')_2^{(l')}$ and $(R)_4^{(l'l'')}$. It follows then $l'=0$ and $F_i'=R$. Hence, G_1 is an abelian group of exponent 2. Now, let G_2^* be the residue class group of G_2 modulo its center. Since G_2^* can be regarded as a subgroup of the direct product of l'' copies of PGL (2, 3) and PGL (2, 3) ($\cong S_4$) is of order $2^3\cdot 3$, G_2^* is a non-trivial 2-group, and hence G_2 is nilpotent. Recalling here that G_2 is indecomposable, we readily see that G_2 is a 2-group.

Concerning the converse, it suffices to prove that if (iii) or (iv) holds then $U(RG)$ is solvable. In fact, if there holds (iii) then $CG=C^{(g)}$, whence we readily see that $RG=(C)_2^{(g)}$ and $U(RG)$ is solvable. Next, if there holds (iv) then $RG_1=R^{(g_1)}$ (g_1 the order of G_1) and $U(RG)=U(RG_1\otimes_R RG_2)$ is solvable.

Remark. Let $R=\mathrm{GF}(3)$. If $g=8$ then $U(RG)$ is solvable. In fact, if G is non-abelian then the commutator subgroup G' of G coincides with the center of G and G/G' is a 2-elementary abelian group of order 4. Accordingly, G possesses four 1 dimensional representations in R, and it is easy to see that $RG=R^{(4)}\oplus(R)_2$. However, in case $g=16$, there exists an indecomposable G such that $U(GR)$ is not solvable: Let G be the group $\langle a,b\rangle$ with defining relations $a^8=1$, $b^2=1$, $b^{-1}ab=a^7$. To be easily seen, G is indecomposable and possesses the irreducible representation $\mathbf{T}$ in R

defined by $\mathbf{T}(a)=\begin{pmatrix}1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & -1 & -1\\ 0 & 0 & 1 & -1\end{pmatrix}$ and $\mathbf{T}(b)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$. The commutor of $\{\mathbf{T}(a), \mathbf{T}(b)\}$ in $(R)_4$ is seen to be $R(\zeta)$, where $\zeta=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$ is a primitive fourth root of 1. Hence, RG contains $(R(\zeta))_2$ as a simple component, and so $U(RG)$ is not solvable.

Theorem 2. *Assume that C is of prime characteristic p and G contains a normal Sylow p-subgroup P. Then, $U(RG)$ is solvable if and only if there holds one of the following*:

(i) *$R=C$ and G is a semi-direct product of P and an abelian group.*

(ii) *$R=(\mathrm{GF}(2))_2$ and $G=P$.*

(iii) *$R=(\mathrm{GF}(3))_2$ and G is a semi-direct product of P and an abelian group of exponent* 2.

(iv) *$R=\mathrm{GF}(3)$ and G is a semi-direct product of P and $G_1\times G_2$, where G_1 is an abelian group of exponent* 2 *and G_2 is a non-abelian indecomposable* 2*-group such that $U(RG_2)$ is solvable.*

Proof. We consider the ring epimorphism $\beta: RG\to RG^*$ defined by $\sum_{\sigma\in G} a_\sigma\sigma \mapsto \sum_{\sigma\in G} a_\sigma\sigma^*$, where $G^*=G/P$ and σ^* is the residue class of σ modulo P. Then, by [2; Theorem 2], Ker β coincides with $J(RG)$. Hence, $U(RG)$ is solvable if and only if so is $U(RG^*)$. Now, our assertion will be immediate by Theorem 1 and Schur-Zassenhaus theorem.

The following corollaries are only selections from Theorems 1 and 2.

Corollary 1. *Let $C\neq$* GF(3).

(a) *Assume that $G\neq 1$ and g is not divisible by the characteristic of C. If $U(RG)$ is solvable then it is nilpotent.*

(b) *Assume that C is of prime characteristic p and G contains a proper normal Sylow p-subgroup P. If $U(RG)$ is solvable then $U(R\cdot G/P)$ is nilpotent, and conversely.*

Corollary 2. *Let $C=$* GF(3).

(a) *Assume that G is not a* 2*-group and g is not divisible by* 3. *If $U(RG)$ is solvable then it is nilpotent.*

(b) *Assume that G contains a normal Sylow* 3-*subgroup P such that G/P is not a* 2-*group. If $U(RG)$ is solvable then $U(R\cdot G/P)$ is nilpotent, and conversely.*

Corollary 3. *Let R be of prime characteristic p, and G a p-group. Then, $U(RG)$ is solvable if and only if $R=C$ or* $(\mathrm{GF}(2))_2$ *or* $(\mathrm{GF}(3))_2$.

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